Real roots of dependent random trigonometric models

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Sommaire

Introduction

Presentation of the model:

- $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ are two independent sequences of a centered stationary **Gaussian** process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}\left[a_k a_l\right] = \mathbb{E}\left[b_k b_l\right] = \rho(k-l) = \int_0^{2\pi} e^{-i(k-l)x} \mu_\rho(dx),$$

μ_ρ: spectral measure.

We assume $\mu_\rho(dx) = \psi_\rho(x)dx + \mu_s$.

- We set $f_n(x) = \dfrac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$,
- $\mathcal{N}(f_n, [a,b]) = \text{Card}\left(x \in [a,b] \mid f_n(x) = 0\right)$.

Independent case

Dunnage (1966): $(a_k)_{k\geq 1}, (b_k)_{k\geq 1}$ i.i.d. $\mathcal{N}(0,1)$ with zero mean and variance one,

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[\mathcal{N}(f_n,[0,2\pi])\right]}{n} = \frac{2}{\sqrt{3}}.$$

☞ In fact, this result holds for any i.i.d. random variable (*Flasche, (2017)*).

Angst, Dalmao, Poly (2017): $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ being μ_ρ-dependent where

$$\mu_\rho(dx) = \psi_\rho(x)dx,$$

with $\psi_\rho \in \mathcal{C}^0\left((0,2\pi)\right)$ and $\inf \psi_\rho > 0$. Then,

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[\mathcal{N}(f_n,[0,2\pi])\right]}{n} = \frac{2}{\sqrt{3}}.$$

☞ In particular, for **long-range correlations** (e.g. increments of FBM) we have universality.

Questions

☞ What happens for a more general spectral measure μ_ρ/density ψ_ρ?
For example: $\psi_\rho = \frac{1}{2a}\mathbb{1}_{[-a,a]}, \; a \in [0, \pi]$.

In particular, what happens when:

- ψ_ρ is not necessarily continuous?
- ψ_ρ is no longer lower-bounded?
- $\lambda\left(\psi_\rho = 0\right) > 0$?

Statement of the results

Lower bound for positive density part

If $\psi_\rho > 0$ a.e. on $[0, 2\pi]$, we always have

$$\liminf_{n \to +\infty} \frac{\mathbb{E}\left[\mathcal{N}(f_n, [0, 2\pi])\right]}{n} \geq \frac{2}{\sqrt{3}}.$$

- Corroborates partially the conjecture (Pirhadi) that $\frac{2}{\sqrt{3}}$ is the least mean number of zeros for f_n.
- $= \frac{2}{\sqrt{3}}$ for ψ_ρ uniformly positively lower-bounded (ADP).

$\psi_\rho > 0$ but no lower-bounded

Theorem

If

$$\begin{cases} d\mu_\rho = \mu_s + \psi_\rho(x)dx \\ \exists \eta > 0, \text{s.t. } \log(\psi_\rho) \in L^{1+\eta}([0,2\pi], dx) \end{cases},$$

then

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[\mathcal{N}(f_n,[0,2\pi])\right]}{n} = \frac{2}{\sqrt{3}}.$$

For $\eta = 0$, $\log(\psi_\rho) \in L^1 \iff (a_k)_{k\geq 1}$ purely 'non-deterministic', i.e. admits a causal representation

$$a_k = \sum_{j=0}^{\infty} c_j N_{k-j}, \quad (c_k)_{k\geq 1} \in \ell^2(\mathbb{N}), \ (N_k)_{k\in\mathbb{Z}} \text{ i.i.d. } \mathcal{N}(0,1).$$

$\lambda(\psi_\rho = 0) > 0$

C.1 : ψ_ρ is piecewise continuous and

$$\{\psi_\rho = 0\} = \bigcup_{i=1}^{p} [a_i, b_i] \cup \bigcup_{j=1}^{q} \{c_j\},$$

C.2 : $\psi_\rho \in \mathcal{C}^{1+\eta}$ s.t. ψ_ρ' is η-Hölder

Theorem

Under **C.1** or **C.2**,

$$\lim_{n \to +\infty} \frac{\mathbb{E}\left[\mathcal{N}(f_n, [0, 2\pi])\right]}{n} = \frac{\lambda\left(\{\psi_\rho = 0\}\right)}{\pi\sqrt{2}} + \frac{\lambda\left(\{\psi_\rho \neq 0\}\right)}{\pi\sqrt{3}}.$$

$\lambda(\psi_\rho = 0) > 0$ vs. $\psi_\rho > 0$ a.e.

Remarks:

- ADP: $\exists \rho \searrow 0$ arbitrarily slow s.t.

$$\lim_n \frac{\mathbb{E}[\mathcal{N}(f_n, [0, 2\pi])]}{n} = \frac{2}{\sqrt{3}}.$$

- Here: $\exists \rho \searrow 0$ arbitrarily fast s.t.

$$\lim_n \frac{\mathbb{E}[\mathcal{N}(f_n, [0, 2\pi])]}{n} \neq \frac{2}{\sqrt{3}}.$$

☞ Universality is not related to the speed of decay of ρ.

<u>Remarks</u>:

- $\lim_n \frac{\mathbb{E}[\mathcal{N}(f_n,[0,2\pi])]}{n}$ only depends on the size of the support of ψ_ρ, not the shape.
- $\left[\frac{2}{\sqrt{3}}, \sqrt{2}\right]$ is the range of attainable values.

Ideas behind the proofs

General framework

$f_n(\cdot)$ is a $\mathcal{C}^1$ process such that

$$\bullet \quad \mathbb{E}[f_n^2(x)] = \frac{1}{n}\sum_{k,\ell=1}^{n} \rho(k-\ell)\cos((k-\ell)x)$$

$$= \underbrace{K_n}_{\text{Fejér kernel}} * \mu_\rho(x) > 0$$

$$\bullet \quad \mathbb{E}[f_n'(x)^2] = \frac{1}{\alpha_n} L_n * \mu_\rho(x),$$

with L_n explicit trigonometric kernel and $\alpha_n = \frac{6}{(n+1)(2n+1)}$.

General framework

Since the field is smooth and non-degenerated, by Kac-Rice formula

$$\mathbb{E}\left[\mathcal{N}(f_n,[0,2\pi])\right] = \frac{1}{\pi}\int_0^{2\pi}\sqrt{\frac{\mathbb{E}[f_n'(x)^2]}{\mathbb{E}[f_n^2(x)]} - \left(\frac{\mathbb{E}\left[f_n(x)f_n'(x)\right]}{\mathbb{E}[f_n^2(x)]}\right)^2}dx.$$

Hence

$$\begin{aligned}
&\frac{\mathbb{E}\left[\mathcal{N}(f_n,[0,2\pi])\right]}{n} \\
&= \frac{1}{\pi}\int_0^{2\pi}\sqrt{\frac{\alpha_n\mathbb{E}[f_n'(x)^2]}{n^2\alpha_n\mathbb{E}[f_n^2(x)]} - \left(\frac{\mathbb{E}\left[f_n(x)f_n'(x)\right]}{n\mathbb{E}[f_n^2(x)]}\right)^2}dx \\
&= \frac{1}{\pi}\int_0^{2\pi}\sqrt{\frac{1}{n^2\alpha_n}\frac{L_n * \mu_\rho(x)}{K_n * \mu_\rho(x)} - \left(\frac{K_n' * \mu_\rho(x)}{2nK_n * \mu_\rho(x)}\right)^2}dx.
\end{aligned}$$

General framework

The properties of the trigonometric kernels give the following a.s. Fejér-Lebesgue convergences:

$$L_n * \mu_\rho(x) \xrightarrow[n\to+\infty]{} \psi_\rho(x),$$

$$K_n * \mu_\rho(x) \xrightarrow[n\to+\infty]{} \psi_\rho(x),$$

$$\frac{1}{n}K_n' * \mu_\rho(x) \xrightarrow[n\to+\infty]{} 0,$$

$$n^2\alpha_n \xrightarrow[n\to+\infty]{} 3.$$

1. What happens when $\psi_\rho > 0$ a.e. ?
2. What happens when ψ_ρ can vanish on an interval?

$$\inf \psi_\rho > 0 \Rightarrow \sqrt{\frac{1}{n^2\alpha_n}\frac{L_n * \psi_\rho(x)}{K_n * \psi_\rho(x)} - \left(\frac{K_n' * \psi_\rho(x)}{2nK_n * \psi_\rho(x)}\right)^2} \xrightarrow[n\to+\infty]{} \frac{1}{\sqrt{3}},$$

by dominated convergence

$$\frac{\mathbb{E}\left[\mathcal{N}(f_n, [0, 2\pi])\right]}{n} \xrightarrow[n\to+\infty]{} \frac{2}{\sqrt{3}}.$$

What about the case of $\psi_\rho > 0$ a.e. & no longer lower bounded?

- Kac-Rice formula does not allow to conclude.
- Another strategy based on Salem-Zygmund like CLT (Jürgen).
- $\log(\psi_\rho) \in L^{1+\eta} \Rightarrow$ uniform integrability condition.

$\mu_\rho(dx) = \psi_\rho(x)dx = \frac{1}{2a}\mathbb{1}_{[-a,a]}(x)dx$

Recall that $\dfrac{\mathbb{E}\left[\mathcal{N}(f_n,[0,2\pi])\right]}{n} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \dfrac{1}{n}\sqrt{I_n(x)}dx$, where

$$\frac{1}{n}\sqrt{I_n(x)} := \sqrt{\frac{1}{n^2\alpha_n}\frac{\overbrace{L_n * \psi_\rho(x)}^{\to\psi_\rho(x)}}{\underbrace{K_n * \psi_\rho(x)}_{\to\psi_\rho(x)}} - \left(\underbrace{\frac{K_n' * \psi_\rho(x)}{2nK_n * \psi_\rho(x)}}_{\to 0}\right)^2}.$$

Strictly inside $[-a,a]$: dominated convergence ($\psi_\rho > 0$) & Fejér -Lebesgue:

$$\frac{L_n * \psi_\rho(x)}{\alpha_n K_n * \psi_\rho(x)} \to \frac{1}{3} \quad ,$$

$\Rightarrow$ Uniformly, $\frac{1}{n}\sqrt{I_n(x)} \to \frac{1}{\sqrt{3}}$.

$\mu_\rho(dx) = \psi_\rho(x)dx = \frac{1}{2a}\mathbb{1}_{[-a,a]}(x)dx$

Recall that $\dfrac{\mathbb{E}\left[\mathcal{N}(f_n,[0,2\pi])\right]}{n} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\frac{1}{n}\sqrt{I_n(x)}dx$, where

$$\frac{1}{n}\sqrt{I_n(x)} := \sqrt{\frac{1}{n^2\alpha_n}\frac{\overbrace{L_n * \psi_\rho(x)}^{\to 0}}{\underbrace{K_n * \psi_\rho(x)}_{\to 0}} - \left(\underbrace{\frac{K_n' * \psi_\rho(x)}{2nK_n * \psi_\rho(x)}}_{\to 0}\right)^2}.$$

More difficult part $\to$ Outside $[-a,a]$: we need 2nd order asymptotics:

$$\frac{L_n * \psi_\rho(x)}{\alpha_n K_n * \psi_\rho(x)} \to \frac{1}{2}.$$

$\Rightarrow$ Uniformly, $\frac{1}{n}\sqrt{I_n(x)} \to \frac{1}{\sqrt{2}}$.

$\mu_\rho(dx) = \psi_\rho(x)dx = \frac{1}{2a}\mathbb{1}_{[-a,a]}(x)dx$

$$\frac{\mathbb{E}\left[\mathcal{N}(f_n,[0,2\pi])\right]}{n} = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{n}\sqrt{I_n(x)}dx, \text{ where}$$

$$\frac{1}{n}\sqrt{I_n(x)} := \sqrt{\frac{1}{n^2\alpha_n}\frac{L_n*\psi_\rho(x)}{K_n*\psi_\rho(x)} - \left(\frac{K_n'*\psi_\rho(x)}{2nK_n*\psi_\rho(x)}\right)^2}.$$

In small neighborhood of $\pm a, 0$, (Prisker-Yeager, 2015), negligible contributions for the roots.

$\mu_\rho(dx) = \psi_\rho(x)dx = \frac{1}{2a}\mathbb{1}_{[-a,a]}(x)dx$

By additivity in Kac-Rice formula,

$$\lim_n \frac{\mathbb{E}\left[\mathcal{N}(f_n, [0, 2\pi])\right]}{n} = \frac{2\pi - 2a}{\pi\sqrt{2}} + \frac{2a}{\pi\sqrt{3}}$$
$$= \frac{\lambda(\{\psi_\rho = 0\})}{\pi\sqrt{2}} + \frac{\lambda(\{\psi_\rho \neq 0\})}{\pi\sqrt{3}}$$

Same method works for finite combinations (Condition C.1).

For more general vanishing locus, we need some regularity.

☞ Non-universality with nodal asymptotics ranging from $\frac{2}{\sqrt{3}}$ to $\sqrt{2}$.

Discrete spectral measure $\mu_\rho = \frac{1}{2}\left(\delta_\alpha + \delta_{-\alpha}\right), \ \alpha \neq 0$

- In general, there is no convergence as $n \to +\infty$.
- $\exists$ a function ℓ^α s.t. $\mathrm{Im}(\ell^\alpha) \subset (\sqrt{2}, 2]$; $\forall \beta \in (0,1)$, for $n \gg 1$ s.t. $n\alpha \notin \pi\mathbb{Z}$,

$$\left| \frac{\mathbb{E}\left[\mathcal{N}(f_n, [0, 2\pi])\right]}{n} - \ell^\alpha(n\alpha \mod \pi) \right| = O\left(\frac{1}{n^\beta (1 - |\cos(n\alpha)|)} \right) + o(1).$$

- In particular, $\forall \epsilon > 0, \forall \ell \in (\sqrt{2}, 2], \exists \alpha \geq 0$ and infinitely many n's

$$\left| \frac{\mathbb{E}\left[\mathcal{N}(f_n, [0, 2\pi])\right]}{n} - \ell \right| \leq \epsilon.$$

☞ Non-universality with nodal asymptotics ranging from $\sqrt{2}$ to 2.

Synthesis

$\mu_\rho = \psi_\rho dx + \mu_s$	$\inf \psi_\rho > 0$	$\log(\psi_\rho) \in L^{1+\eta}$	$\mu_s = 0$ $\lambda(\{\psi_\rho = 0\}) =: a > 0$	$\mu_\rho = \frac{\delta_\alpha + \delta_{-\alpha}}{2}$ $\alpha \notin \pi\mathbb{Q}$
$\frac{\mathbb{E}[\mathcal{N}(f_n,[0,2\pi])]}{n}$	$\to \frac{2}{\sqrt{3}}$	$\to \frac{2}{\sqrt{3}}$	$\to \frac{2\pi - 2a}{\pi\sqrt{2}} + \frac{2a}{\pi\sqrt{3}}$	$\not\to$ $\sqrt{2}\liminf < \limsup \leq 2$

Thank you for your attention.